Math 565: Functional Analysis
Lecture 06

Examples (continued).

(b) Let (X,μ) and (Y,ν) be σ-finite measure spaces and let $K: X\times Y \to \mathbb{C}$ be $\mu\times\nu$-measurable. Suppose $\exists\, C_1, C_2 > 0$ such that $\|K(\cdot,y)\|_{L^1(\mu)} \le C_1$ for a.e. $y\in Y$ and $\|K(x,\cdot)\|_{L^1(\nu)} \le C_2$ for a.e. $x \in X$. Then for each $1\le p\le\infty$, $T_K: L^p(\nu)\to L^p(\mu)$ define by
$$T_K f(x) := \int K(x,y) f(y)\, d\nu(y)$$
is bdd linear transformation with $\|T_K\| \le C_1^{\frac{1}{p}} C_2^{\frac{1}{q}}$, where q is the conj. exp. of p. This was shown in HW.

Important special case: convolutions. Let $g \in L^1(\mathbb{R}^d,\lambda)$ and set $K(x,y) := g(x-y)$. Then $\|K(\cdot,y)\|_1 = \int |g(x-y)|\, d\lambda(x) = \int |g(x)|\, d\lambda(x) = \|g\|_1 < \infty$, by the translation invariance of λ. Similarly,
$$\|K(x,\cdot)\|_1 = \int |g(x-y)|\, d\lambda(y) \overset{\text{transl. inv.}}{=} \int |g(-y)|\, d\lambda(y) \overset{\text{invariance under mult by } -1}{=} \int |g(y)|\, d\lambda(y) = \|g\|_1 < \infty.$$
So $C_1 = C_2 = \|g\|_1$, hence we have that the transformation, for each $1\le p\le\infty$, $T_K: L^p(\mathbb{R}^d,\lambda)\to L^p(\mathbb{R}^d,\lambda)$ given by
$$T_K f(x) := \int g(x-y) f(y)\, d\lambda(y)$$
is bdd linear and $\|T_K\| \le C_1^{\frac{1}{p}} C_2^{\frac{1}{q}} = \|g\|_1^{\frac{1}{p}+\frac{1}{q}} = \|g\|_1$.
This transformation is called the convolution by g.
We denote by $f * g(x) := \int g(x-y) f(y)\, d\lambda(y)$ and call this the convolution of f and g. Note that $f*g = g*f$, by the translation and mult. by -1 invariance of λ.

(c) Let $D: C^1[0,1] \to C[0,1]$ be given by $f \mapsto f'$. If $C^1[0,1]$ is equipped with the norm $\|f\| := \max(\|f\|_u, \|f'\|_u)$, then $\|Df\|_u = \|f'\|_u \le \|f\|$, hence $\|D\|\le 1$, in fact $\|D\| = 1$ because of $f(x) = x$. But if we took just the uniform norm on $C^1[0,1]$ (in particular, it wouldn't be a Banach space), then D is unbdd. Indeed, Let $f_n(x) := \sin(2\pi n x)$, then $Df_n(x) = 2\pi n \cos(2\pi n x)$ so $\|Df_n\| = 2\pi n \underset{n\to\infty}{\longrightarrow} \infty$ while $\|f_n\| = 1$ for all n.

(d) Nonexample. Let X be any ∞-dim normed vector space (e.g. $\ell^p(\mathbb{N})$, $L^p(Y,\nu)$, $\mathrm{span}(e_n : n \in \mathbb{N})$ with $e_n \in \ell^2(\mathbb{N})$ standard, i.e. $e_n := \mathbb{1}_{\{n\}}$).
By Zorn's lemma, taking a maximal independent set, there is a linear basis B for X, called Hamel basis. Since B is infinite, $\exists \{b_n\} \subseteq B$ of pairwise distinct vectors. To define a linear transf. it is enough to define it on B and extended uniquely. Define $T: X \to X$ by setting $Tb_n := n b_n$ and $T(b) = 0$ or anything else for $b \in B \setminus \{b_n\}$. Then T is linear but unbdd because $\|Tb_n\| = n\|b_n\|$.

(e) An example of bdd bijective lin. trans. with unbdd inverse. Let $X := \mathrm{span}(e_n : n \in \mathbb{N})$, where $e_n := \mathbb{1}_{\{n\}} = (0,0,\dots,0,\overset{n}{1},0,0,\dots)$. For any $1 \le p \le \infty$, X is a (non-closed) subspace of $\ell^p(\mathbb{N})$ exactly of vectors which have only finitely many nonzero coordinates. (In fact, X is dense in ℓ^p for $1 \le p < \infty$.)
Define a linear $T: X \to X$ uniquely by $T(e_n) := \frac{1}{n} e_n$. Then for any $v \in X$, $v = \sum_{n \le N} \alpha_n e_n$ so $\|Tv\|_p \le \|v\|_p$ so T is bdd. However, $T^{-1} e_n = n e_n$ hence T^{-1} is unbdd since $\|T^{-1} e_n\|_p = n \cdot \|e_n\|_p$.

$$Tv = T\Big(\sum_{n \le N} \alpha_n e_n\Big) = \sum_{n \le N} \alpha_n T e_n = \sum_{n \le N} \frac{1}{n} \alpha_n e_n.$$

Bounded linear functionals.

When a linear transformation is $\mathbb{C}$-valued, it is call a linear functional. Using Zorn's lemma, as in Example (d) above, we can get pathological linear functionals, so we only focus on bdd linear functionals. For a normed vector space X, denote by

$$X^* := B(X, \mathbb{C})$$

the space of all bdd linear functionals on X and call this the dual of X.

Examples.

(a) Let $1 \le p \le \infty$ and let q be the conj. exp. of p. Fix any measure space (X, μ). Fix $g \in L^q := L^q(X, \mu)$. Define $I_g : L^p \to \mathbb{C}$ by

$$f \mapsto \int f g \, d\mu.$$

Then by Hölder, $|I_g f| = |\int f g \, d\mu| \le \|fg\|_1 \le \|f\|_p \|g\|_q$, so $\|I_g\| \le \|g\|_q$. By taking $f := \frac{|g|^{q-1} \overline{\mathrm{sgn}\, g}}{\|g\|_q^{q-1}}$, we see that (as before), $I_g f = \int f g \, d\mu =$

$$= \frac{1}{\|g\|_q^{q-1}} \int |g|^q \, d\mu = \|g\|_q^q / \|g\|_q^{q-1} = \|g\|_q, \text{ so } \|I_g\| = \|g\|_q.$$

These I_g are bdd linear functionals and the map $L^q \mapsto (L^p)^*$ by $g \mapsto I_g$ is a linear isometry. We will show that for $1 \le p < \infty$, this is actually surjective, i.e. an isometric isomorphism.

(b) Fix a measurable space $(X, \mathcal{B})$. Fix a (finite) complex measure ρ on $(X, \mathcal{B})$. Recall that putting $\mu = |\mathrm{Re}\,\rho| + |\mathrm{Im}\,\rho|$, we have $\rho \ll \mu$ so $\forall B \in \mathcal{B}$,

$$\rho(B) = \int \frac{d\rho}{d\mu} \, d\mu$$

and $$|\rho|(B) = \int \left| \frac{d\rho}{d\mu} \right| d\mu = \sup_{|\varphi| \le 1} \left| \int_B \varphi \, d\rho \right|, \text{ so } \|\rho\|_{TV} = \sup_{|\varphi| \le 1} \left| \int_X \varphi \, d\rho \right|. \quad (\star)$$

- Let $B(X, \mathcal{B})$ denote the space of bdd $\mathcal{B}$-measurable functions with uniform norm. Then $B(X, \mathcal{B})$ is a closed subspace of $\mathcal{B}(X)$ of all bdd functions with the uniform norm (since $\mathcal{B}$-meas. functions are closed under limits), hence $B(X, \mathcal{B})$ is a Banach space. Define $I_\rho : B(X, \mathcal{B}) \to \mathbb{C}$. This is a linear
$$f \mapsto \int f \, d\rho$$
functional and by $(\star)$, we have
$$\|I_\rho\| = \sup_{\|\varphi\|_u \le 1} \left| \int \varphi \, d\rho \right| = \sup_{|\varphi| \le 1} \left| \int \varphi \, d\rho \right| = \|\rho\|_{TV} < \infty,$$ and this sup is achieved.

Thus, the space $M(X, \mathcal{B})$ of all complex measures on $(X, \mathcal{B})$ isometrically embeds into $B(X,\mathcal{B})^*$.

- Now let X be a Polish space (e.g. $[0,1]$, $\mathbb{R}$, $(0,1)$, $2^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$, $\mathbb{R}^{\mathbb{N}}$) and $\mathcal{B} := \mathcal{B}(X) :=$ the Borel sets. In particular all finite measures on X are regular (i.e. approximable from above by open and from below by closed sets) and tight (i.e. approximable from below by compact sets). Restricting I_ρ to $BC(X) :=$ the space of bdd continuous functions, we still have that I_ρ is linear and $\|I_\rho\| \le \|\rho\|_{TV}$. We show that actually $\|I_\rho\| = \|\rho\|_{TV}$ by approximating every $\mathcal{B}$-measurable $|\varphi| \le 1$ with bdd a bdd cont. fnct. in $L^1(X, |\rho|)$. Indeed, using regularity, one can show that $BC(X)$ is dense in $L^1(X,|\rho|)$ in the L^1-norm (HW), so letting $|\varphi| \le 1$ be a $\mathcal{B}$-measurable function with $|I_\rho \varphi| = \|\rho\|_{TV}$ and $\varepsilon > 0$, take $\tilde{\varphi} \in BC(X)$ with $\|\varphi - \tilde{\varphi}\|_{L^1(|\rho|)} < \varepsilon$, so $|I_\rho \tilde{\varphi}| \approx_\varepsilon |I_\rho \varphi| = \|\rho\|_{TV}$.
Thus, again, $M(X,\mathcal{B})$ isometrically embeds into $BC(X)^*$.

<u>Caution.</u> Unfortunately, even $BC(X)^*$ is much larger than $M(X,\mathcal{B})$, it also contains all finitely additive measures on $\mathcal{B}$, including all ultrafilters on $\mathcal{B}$.

- To get that $M(X,\mathcal{B})$ is all of the dual, we need to take a smaller closed subspace of $BC(X)$, namely,
$$C_0(X) := \{f \in BC(X) : f \text{ vanishes at } \infty\},$$
where we say that $f : X \to \mathbb{C}$ *vanishes at* ∞ if $\forall \varepsilon > 0$ $\{|f| \ge \varepsilon\}$ is compact. We need to further assume that X is locally compact. Then indeed, $M(X,\mathcal{B}) \cong C_0(X)^*$ and this is known as the *Reisz representation theorem*.